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P. R. Bandyopadhyay
Associate Editor

Layer-by-Layer Analysis of a Simply Supported Thick Flexible Sandwich Beam

Hemendra Arya,* R. P. Shimpi,† and N. K. Naik†
Indian Institute of Technology Bombay, Mumbai 400 076, India

Introduction

MODERN sandwich structures consist of stiff faces, often unsymmetrical, and plastic foam as core, which is flexible in nature. To analyze such a structure, a theory is needed that can account for the transverse flexibility of the core and shear deformation. Traditionally, sandwich structures are analyzed using Timoshenko theory, in which the shear stress distribution is assumed to be constant across the thickness and the core is assumed to be rigid in transverse direction (see Ref. 1). A refinement to such a theory is a higher-order displacement field, where shear stress distribution is not assumed constant. Such refinements can account for shear deformation very easily, but transverse flexibility is not accounted for. A further refinement is to enhance the core analysis where transverse flexibility can be accounted for. Frostig et al.[2] developed a sandwich beam theory that accounts for core flexibility. This theory is extended by Swanson[3] for multiple supports and overhang. Recently, Pai[4] and Perel and Palazotto[5] developed finite element for flexible core sandwich structures. A good review on sandwich structures is given in Ref. 6.

In the present work, a simply supported sandwich beam of isotropic layers is considered. The two thin stiff faces are connected to a flexible core. The thin faces are analyzed using classical beam theory, where the assumptions of the plane section remaining plane and constant thickness are applicable. The thick core is analyzed using a more involved solution, that is, two-dimensional elasticity theory under plane stress conditions.[7] This two-dimensional solution, being general in nature, will account for transverse flexibility of the core, as well as shear deformation. This technique reduces the complexity of the solution as compared to if all of the layers are analyzed a using two-dimensional solution for such a problem.

In the present formulation, in-plane and transverse displacement boundary conditions are satisfied at the interface. The shear stress at the interface is modeled as moment and an in-plane force in the face.

For comparison, results of a solution is also obtained for the sandwich beam when all of the three layers are analyzed using two-dimensional elasticity under plane stress conditions. Present results are also compared with a conventional higher-order theory and a layer-by-layer theory referred to as trigonometric shear deformation theory–equivalent single layer (TSDT-ESL) and TSDT–zig zag (TSDT-ZZ). The displacement field and governing equations for these refined theories are given in Ref. 8.

Formulation of the Problem

A schematic diagram of the sandwich beam is shown in Fig. 1a. Let Young's modulii of the top face be E_t, the core be E_c, the bottom face be E_b, and Poisson's ratio of the core be ν_c. The thickness of the top face is t_t, that of the core is h_c, and that of the bottom face is t_b. The beam width is b.

The top and bottom faces are taken as a beam connected to an elastic core. The various stresses acting on the skins and core are shown in Fig. 1b. The shear stress at the interface is modeled as an in-plane load and a moment in the skin for the analysis as shown in Fig. 1c.

Let the lateral loading at the top of the top face be q and

$$q = \sum_{m=1}^{\infty} E_m \sin\alpha x, \qquad E_m = \frac{2}{l}\int_0^l q \sin\alpha x \, \mathrm{d}x \tag{1}$$

The stresses acting on the core are represented in series form, at the top of the core

$$\sigma_y = \sum_{m=1}^{\infty} -B_m \sin\alpha x, \qquad \tau_{xy} = \sum_{m=1}^{\infty} C_m \cos\alpha x \tag{2}$$

and at the bottom of the core

$$\sigma_y = \sum_{m=1}^{\infty} -A_m \sin\alpha x, \qquad \tau_{xy} = \sum_{m=1}^{\infty} D_m \cos\alpha x \tag{3}$$

where $\alpha = m\pi/l$ and l is the length of the beam and E_m is constant and depends on loading on the top face. A_m, B_m, C_m, and D_m are unknowns.

Solution for the Top Face

Figure 1c shows the loads acting on the top face. The lateral deflection v_t of the top face is assumed in terms of unknown constant S_m as

$$v_t = \sum_{m=1}^{\infty} S_m \sin\alpha x \tag{4}$$

From Fig. 1c, note that lateral loading and distributed moment in the top face is

$$q_y = \sum_{m=1}^{\infty} \sin\alpha x (E_m - bB_m), \qquad T_x = \frac{bt_t}{2}\sum_{m=1}^{\infty} C_m \cos\alpha x \tag{5}$$

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*Research Associate, Aerospace Engineering Department, Powai; currently Research Scientist, Center for Aerospace Systems Design and Engineering, Indian Institute of Technology Bombay, Mumbai 400 076, India; arya@aero.iitb.ac.in.

†Professor, Aerospace Engineering Department, Powai.

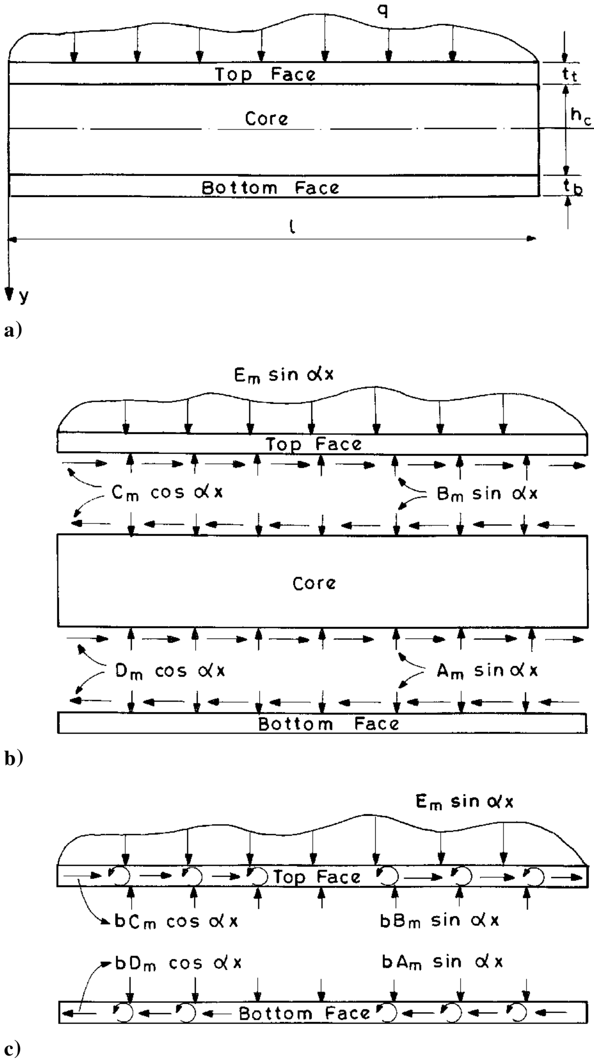

Fig. 1 Schematic of sandwich beam and loading on different elements.

Substitute the expressions for q_y and T_x in Eq. (A3) (see Appendix). The governing equation for the top face is obtained as

$$\sum_{m=1}^{\infty} E_t I_t \alpha^4 S_m + \frac{bt_t}{2}\alpha C_m - (E_m - bB_m) = 0 \tag{6}$$

In Eq. (6), I_t is moment of inertia of the top face.

From Fig. 1c note that axial loading in the top face is

$$p_x = b\sum_{m=1}^{\infty} C_m \cos\alpha x \tag{7}$$

When Eqs. (4) and (7) are used in Eq. (A8), the total axial displacement u_t in the top face due to bending and in-plane load is obtained as

$$u_t = \sum_{m=1}^{\infty}\left\{\frac{C_m}{E_t t_t \alpha^2} - y\alpha\, S_m\right\}\cos\alpha x \tag{8}$$

Solution for the Core

The core is assumed to be an in-plane stress condition. The solution is obtained by using a stress function satisfying the biharmonic equation[7]:

$$\nabla^2\nabla^2\phi = 0 \tag{9}$$

The stress function ϕ is

$$\phi = \sum_{m=1}^{\infty} \sin\alpha x(C_{1m} ch\alpha y + C_{2m} sh\alpha y + C_{3m} ych\alpha y + C_{4m} ysh\alpha y) \tag{10}$$

C_{1m}, C_{2m}, C_{3m}, and C_{4m} are constants to be obtained from the boundary conditions, that is, the stress boundary condition on the top and bottom of the core [Eqs. (2) and (3)].[7] The stresses acting on the core are shown in Fig. 1b.

Stresses are given in terms of stress function as

$$\sigma_y = \frac{\partial^2\phi}{\partial x^2}, \qquad \sigma_x = \frac{\partial^2\phi}{\partial y^2}, \qquad \tau_{xy} = -\frac{\partial^2\phi}{\partial x\partial y} \tag{11}$$

In general, displacements are given as

$$v_c = \frac{1}{E_c}\left\{\int_y \frac{\partial^2\phi}{\partial x^2}\,\mathrm{d}y - \nu_c\frac{\partial\phi}{\partial y} + g(x)\right\}$$

$$u_c = \frac{1}{E_c}\left\{\int_x \frac{\partial^2\phi}{\partial y^2}\,\mathrm{d}x - \nu_c\frac{\partial\phi}{\partial x} + f(y)\right\} \tag{12}$$

where v_c and u_c are lateral and in-plane displacement.

Solution for the Bottom Face

Figure 1c shows the loads acting on the bottom face. The lateral deflection v_b of the bottom face is assumed in terms of the unknown constant T_m as

$$v_b = \sum_{m=1}^{\infty} T_m \sin\alpha x \tag{13}$$

From Fig. 1c note that lateral loading and distributed moment in the bottom face is

$$q_y = \sum_{m=1}^{\infty} bA_m \sin\alpha x, \qquad T_x = \frac{bt_b}{2}\sum_{m=1}^{\infty} D_m\cos\alpha x \tag{14}$$

Substitute the expressions for q_y and T_x in Eq. (A3). The governing equation for the bottom face is obtained as

$$\sum_{m=1}^{\infty} E_b I_b \alpha^4 T_m + \frac{bt_b}{2}\alpha D_m - bA_m = 0 \tag{15}$$

In Eq. (15), I_b is moment of inertia of the bottom face.

From Fig. 1c note that axial loading in the bottom face is

$$p_x = -b\sum_{m=1}^{\infty} D_m\cos\alpha x \tag{16}$$

When Eqs. (13) and (16) are used in Eq. (A8), the total axial displacement u_b in the bottom face due to bending and in-plane load is obtained as

$$u_b = \sum_{m=1}^{\infty}\left\{-\frac{D_m}{E_b t_b\alpha^2} - y\alpha T_m\right\}\cos\alpha x \tag{17}$$

Derivation of Governing Equations for Sandwich Beam

The governing equations for the top face, core, and bottom face that have been obtained in the preceding sections will be utilized to obtain the governing equations for a sandwich beam. The governing equations are obtained by comparing the in-plane displacement and lateral displacement at the interface and using the beam bending equations (6) and (15).

Comparing v at $y = +h_c/2$ and using Eqs. (12) and (13) one gets

$$\frac{1}{E_c}\left\{\int_y \frac{\partial^2\phi}{\partial x^2}\,\mathrm{d}y - \nu_c\frac{\partial\phi}{\partial y}\right\}\Bigg|_{y=+h_c/2} = T_m \sin\alpha x \tag{18}$$

Comparing u at $y = +h_c/2$ and using Eqs. (12) and (17), one gets

$$\frac{1}{E_c}\left\{\int_x \frac{\partial^2\phi}{\partial y^2}\,\mathrm{d}x - \nu_c\frac{\partial\phi}{\partial x}\right\}\Bigg|_{y=+h_c/2}$$

$$= \left\{-\frac{D_m}{E_b t_b \alpha^2} - \frac{h_c}{2}\alpha T_m\right\}\cos\alpha x \tag{19}$$

Comparing v at $y = -h_c/2$ and using Eqs. (4) and (12), one gets

$$\frac{1}{E_c}\left\{\int_y \frac{\partial^2\phi}{\partial x^2}\,\mathrm{d}y - \nu_c\frac{\partial\phi}{\partial y}\right\}\Bigg|_{y=-h_c/2} = S_m \sin\alpha x \tag{20}$$

Comparing u at $y = -h_c/2$ and using Eqs. (8) and (12), one gets

$$\frac{1}{E_c}\left\{\int_x \frac{\partial^2\phi}{\partial y^2}\,\mathrm{d}x - \nu_c\frac{\partial\phi}{\partial x}\right\}\Bigg|_{y=-h_c/2}$$

$$= \left\{\frac{C_m}{E_t t_t \alpha^2} + \frac{h_c}{2}\alpha S_m\right\}\cos\alpha x \tag{21}$$

The governing equations (6), (15), and (18–21) contain six unknown constants (A_m, B_m, C_m, D_m, S_m, and T_m). These constants are obtained for each m. When these constants are used, displacements and stresses can be obtained in the core and skins.

Numerical Results and Discussion

The function v_b, v_t, and ϕ chosen satisfy the following boundary conditions at $x = 0, l$: $\sigma_x = 0$,

$$\int \sigma_x y\,\mathrm{d}y = 0$$

and $v = 0$. Numerical results are presented for different core elasticity modulus and uniform loading on the top of the beam. The material properties considered are $E_t/E_c = 1, 10, 100,$ and 500; $\nu_c = 0.3$; and $E_t = E_b$. The geometry of the beam is $h_c = 0.9h, t_t = t_b = 0.05h$, and $l/h = 20$, where h is total thickness of the beam.

The material properties range from an isotropic beam ($E_t/E_c = 1.0$) to a very flexible core ($E_t/E_c = 500$). The results obtained from the present theory are compared with the two-dimensional elasticity solution under plane stress for a layered beam of isotropic material. Results are also compared with displacement based higher-order theory of two types, that is, conventional equivalent single layer (TSDT-ESL) and an advanced layerwise zig-zag theory (TSDT-ZZ).[8] In the case of TSDT-ESL, the transverse shear stress is obtained using an equilibrium equation approach which is a standard method in such theories.

Here, $\bar{v} = E_t h^3 v(l/2, 0)/ql^4$, $\bar{u} = E_t u(0, h/2)/hq$, and $\bar{\tau}_{xy} = \tau_{xy}(0, 0)/q$.

Table 1 shows the results for sandwich beam subjected to uniformly distributed load (UDL) on the top face. Note that, as expected, TSDT-ESL predictions are not good for flexible core, that is, $E_t/E_c = 500$.

To demonstrate the capability of the present model, a thick beam is considered in which $E_t/E_c = 500$ and $l/h = 10$. The in-plane displacement $\bar{u}$, when it is subjected to a UDL, is shown in Fig. 2. Note that present method's predictions match very well with the beam solution when all three of the layers are analyzed using a two-dimensional elasticity solution under plane stress conditions. TSDT-ZZ is able to capture the kink but not the compression effect. TSDT-ESL is not adequate for the geometry, material property, and loading considered here.

It is known that beam bending is close to the plane stress problem, and it is more appropriate when the beam is narrow.[7] Note that, in the literature, the beam bending results are compared with cylindrical bending of the plate problem, which is a direct reduction of a three-dimensional solution to a two-dimensional solution by considering one of the dimensions in the plane of the plate as infinite. Cylindrical bending of plate is a plane strain problem.[9]

Table 1 Sandwich beam subjected to UDL, $l/h = 20$

Theory	$\bar{v}$	$\bar{u}$	$\bar{\tau}_{xy}$
	$E_t/E_c = 1$		
Two-dimensional	0.1571	1798.0	14.81
Present	0.1570	1802.0	14.83
TSDT-ESL	0.1572	1801.0	14.78
TSDT-ZZ	0.1572	1801.0	15.38
	$E_t/E_c = 10$		
Two-dimensional	0.4625	5218.0	11.69
Present	0.4629	5120.0	11.69
TSDT-ESL	0.4621	5230.0	11.58
TSDT-ZZ	0.4627	5230.0	12.02
	$E_t/E_c = 100$		
Two-dimensional	0.6423	6384.0	10.54
Present	0.6431	6398.0	10.54
TSDT-ESL	0.6237	6430.0	10.44
TSDT-ZZ	0.6426	6489.0	10.64
	$E_t/E_c = 500$		
Two-dimensional	0.9776	6256.0	10.34
Present	0.9787	6290.0	10.34
TSDT-ESL	0.7727	6470.0	10.34
TSDT-ZZ	0.9778	6284.0	10.38

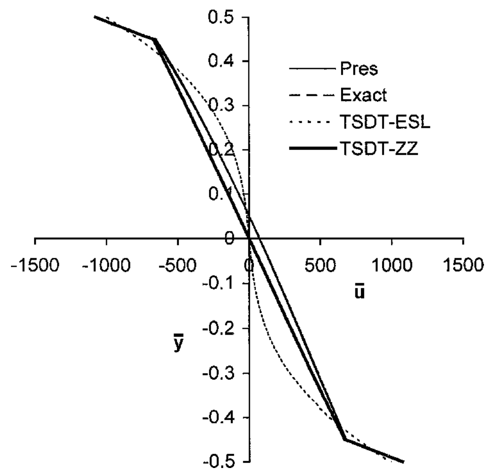

Fig. 2 Comparison of in-plane displacement $\bar{u}$ for $E_t/E_c = 500$, $l/h = 10$, UDL.

Conclusions

The present problem considered is a three-isotropic-layer beam. If solved using a two-dimensional elasticity solution under plane stress conditions, 12 unknowns have to be evaluated. In the present problem, six unknowns are evaluated by simplifying the analysis of the top and bottom faces where it is appropriate, and the core is analyzed by the more rigorous solution where it is required. It is a pseudoexact solution for a sandwich beam.

Appendix: Governing Equation for a Beam Element

Figure A1 shows a beam element subjected to various forces and moments. The govening equation for such a beam element is

$$\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial T_x}{\partial x} + q_y = 0 \tag{A1}$$

where

$$M_x = -EI\frac{\partial^2 v}{\partial x^2} \tag{A2}$$

where E and I are elastic modulus and moment of inertia of the beam element.

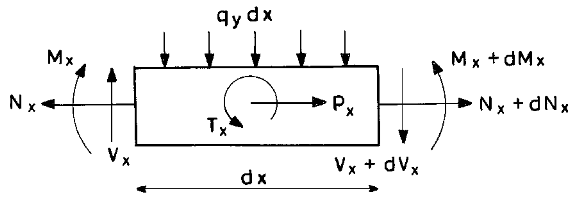

Fig. A1 Beam element subjected to different loads.

The governing equation for the beam subjected to lateral loading and distributed moment can be obtained by substituting Eq. (A2) in Eq. (A1),

$$-EI\frac{\partial^4 v}{\partial x^4} + \frac{\partial T_x}{\partial x} + q_y = 0 \tag{A3}$$

In-plane displacement due to bending of the beam

$$u^b = -y\frac{\partial v}{\partial x} \tag{A4}$$

Equilibrium of in-plane forces gives

$$\frac{\partial N_x}{\partial x} + p_x = 0 \tag{A5}$$

where p_x is distributed in-plane load.

For a beam of width b and thickness t,

$$N_x = bt\sigma_x, \qquad \sigma_x = E\epsilon_x = E\frac{\partial u_a}{\partial x} \tag{A6}$$

where u^a is in-plane displacement due to axial load. Sustitute Eq. (A6) in Eq. (A5) and integrate:

$$u^a = \frac{1}{btE}\int_x\int_x -p_x\,\mathrm{d}x\,\mathrm{d}x \tag{A7}$$

The total in-plane displacement due to bending and axial force is

$$u = u^b + u^a = -y\frac{\partial v}{\partial y} + \frac{1}{btE}\int_x\int_x -p_x \tag{A8}$$

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A. N. Palazotto
Associate Editor

Optimal Fiber Angles to Resist the Brazier Effect in Orthotropic Tubes

Luca S. Cecchini* and Paul M. Weaver†
University of Bristol,
Bristol, England BS8 1TR, United Kingdom

Introduction

As a thin-walled, circular, cylindrical shell is subjected to bending deflections, it will tend to ovalize according to the Brazier[1] effect. In doing so, the diminishing cross-sectional second moment of area of the tube reduces the flexural stiffness of the structure. This Technical Note details the ply configurations of composite tubes that maximize the critical failure (Brazier[1]) moment, where the reduced second moment of area is no longer able to sustain the applied moment.

Vlasov's[2] semimembrane theory assumes that longitudinal bending of a tube is resisted principally by A_{11} (longitudinal stiffness) terms and cross-sectional deformation by D_{22} (circumferential bending stiffness). This dependency is confirmed both by Kedward,[3] who expanded Brazier[1] analysis to include orthotropic shells, as well as by Harursampath and Hodges.[4] They found the critical failure load of a thin-walled cylinder under bending to be

$$M_{\mathrm{cr}} = 3.42a\sqrt{A_{11}\cdot D_{22}} \tag{1}$$

$$M_{\mathrm{cr}} = 4.223a\sqrt{A_{11}\cdot D_{22}} \tag{2}$$

respectively (where a is the tube radius). Despite coefficient differences, both models confirm that a maximization of limit moment must maximize $A_{11}\cdot D_{22}$.

This Technical Note provides an analytical optimization of shell composition with respect to the critical Brazier[1] moment, which will be useful in the design of structures where material failure and local buckling are not likely to occur.

Ply Configuration

The A_{11} term is a tensile/compressive stiffness and, thus, dependent solely on cross-sectional area of the plies. As such, it is not a function of stacking order, but only of the average longitudinal stiffness of plies, and is maximized by including as many 0-deg plies as possible. Conversely, the transverse shell bending stiffness D_{22} is stacking order dependent, being a function of the shell wall second moment of area. The maximization of this term is dependent on the placement of circumferential (90-deg) plies far from the shell wall's neutral axis.

It, therefore, appears intuitive that in highly unidirectional composites, the optimal layup will consist of 90-deg plies sandwiching a 0-deg layer. To test this hypothesis, a generalized orthotropic, symmetric three-layer configuration of balanced ($\pm\theta$) plies, $[\pm\theta(O), \pm\theta(M)]_s$ (Fig. 1) is studied to determine the optimum ply angle of each layer. The validity of only three layers will be analyzed subsequently, and the optimum relative layer thicknesses will be found.

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*Graduate Student, Department of Aerospace Engineering, Queen's Building; luca.cecchini@bristol.ac.uk. Student Member AIAA.

†Lecturer, Department of Aerospace Engineering, Queen's Building; paul.weaver@bristol.ac.uk. Member AIAA.